Nuclear Physics B61 (1973) 253–273. North-Holland Publishing Company

WEAK INTERACTIONS AT HIGH ENERGIES AND COMPLEX ANGULAR MOMENTA

A.A. ANSELM and V.N. GRIBOV
Leningrad Nuclear Physics Institute, USSR

Received 15 December 1972

Abstract: It has been suggested that weak interactions at high energies, $Gs > 1$ (G is the Fermi coupling constant, $\sqrt{s}$ is the c.m.s. energy), become strong so that the partial waves can reach their unitary limit. This hypothesis is the opposite one to the assumption that there exists some special mechanism (e.g. the intermediate W-meson) which cuts off the increase of the partial waves at comparatively small energies $s \ll G^{-1}$, in which case the interaction could remain weak up to very high energies. Elastic lepton scattering and some other leptonic reactions ($e^+ + e^- \rightleftharpoons \mu^+ + \mu^-$) are considered by means of complex angular momentum theory provided $Gs \gg 1$. The main difference with the usual strong interactions consists in the possibility of the exchange of massless particles in the cross t-channel. This results (i) in the appearance of the singularity at $t = 0$ in Regge trajectories and residues and (ii) in the condensation of the Regge poles at $t \to 0$ in the j-plane along the imaginary axis $\mathrm{Re}\, j = 0$. For inelastic processes such as $e^+ + e^- \to \mu^+ + \mu^-$ the latter singularities could be the leading ones and the elastic cross section has then a specific oscillating behaviour at small momentum transfer. In the case of elastic lepton scattering due to s-channel unitarity there must exist some additional singularity in the right-half j-plane at $t = 0$. The total leptonic cross section satisfies therefore the inequality $\sigma_{\mathrm{tot}} > \mathrm{const} \cdot s^{-1}$ when $s \to \infty$.

1. Introduction

It is well-known that the phenomenological description of weak interactions at small energies by perturbation theory with the four-fermion Lagrangian should be changed at energies $s \lesssim G^{-1}$, where G is the Fermi coupling constant. At $Gs \sim 1$ the partial S- and P-waves as calculated by perturbation theory saturate their unitary limit and the application of the perturbation theory at higher energies would violate unitarity. Apparently there are two possibilities for "unitarization" of the theory. One of them is that the application of the local four-fermion Lagrangian becomes invalid at some energy which is essentially *smaller* than the unitary limit. If at this energy the partial waves stop growing (or there remains only the logarithmic increase), the energies $s \sim G^{-1}$ are irrelevant to the problem and the partial waves can remain small up to very high energies. The typical situation of that kind arises if one tries to decribe the weak interaction together with the electromagnetic one by the introduction of a triplet of vector mesons. The mass of the intermediate charged meson is then the characteristic energy at which the increase of the partial waves is cut off and up to logarithmicly high energies they can be of order of $\lesssim \alpha = 1/137$. Note, that even here, because of the logarithmic increases, the interaction becomes eventually strong, but only at academic energies $\alpha \ln s/M_W^2 \sim 1$ (or even $\alpha^2 \ln s/M_W^2 \sim 1$). It is obvious that in such an approach one must construct the renormalizable theory. It is just renormalizability that provides the absence of the power increase of partial waves. Recently essential progress has been achieved in the construction of the renormalizable theory of weak interactions [1]. However, it seems to us that in some way this theory is not an economical one. The point is that in any theory of such a type the introduction of a new mass is required (e.g., the mass of the intermediate boson) whereas at energy $s \sim G^{-1}$ the cut off must take place simply because of unitarity. Therefore, though it is impossible to exclude the version with the cut off at some energy smaller than at the unitary limit, we shall discuss in what follows the other possibility, namely, the unitarization of the theory at $s \sim G^{-1}$ when the partial waves are already of order of unity.

The appearance of the power increase in weak interactions at small energies is in fact of a purely kinematical character. Indeed, the amplitude of the lepton-lepton scattering is proportional to Gs due to the necessity of vanishing of the lepton-antilepton scattering amplitude in the backward direction. This, in turn, is necessary because of helicity conservation. The situation with weak interactions may be considered then as follows. The weak interactions are, generally speaking, strong, i.e. the partial waves can reach their unitary limit. At low energies $s < G^{-1}$, however, they are small because of the kinematical requirement of vanishing of the amplitude at $s = 0$. The situation could be similar to that for the strong interactions for π-π scattering in the soft-pion limit. If the PCAC hypothesis is right then the π-π scattering amplitude must be proportional at small energies to some combination of s, t and u (when $m_\pi = 0$) and, hence, must increase with the increase of the invariants. However, at energies of, say, 0.5 GeV this increase stops and we have the usual intricate picture of strong interactions. The main difference from the weak interactions consists, of course, in the fact that in the latter case the "low-energy domain" survives up to energies of order of $G^{-\frac{1}{2}}$, i.e. up to 300–1000 GeV. (A more accurate estimation of the boundary, where weak interactions should be considered as strong, is made by estimation of the energy at which an appropriate partial wave saturates its unitary limit [2]. We shall discuss this in future in more detail.)

Whereas for theories where the interaction remains weak for practically all energies the main problem is the choice of the Hamiltonian, for a theory in which the interaction becomes strong at $Gs > 1$, the usual strong-interaction methods are rather to be applied. At high energies, which means here $Gs \gg 1$, it is reasonable to try to use the complex angular momentum theory. This is what is done in the present paper. The distinguishing feature of the case is the possibility of the exchange of massless particles (neutrino and also electrons and μ-mesons if $t \gg m_\mu^2, m_e^2$) in the cross t-channel. In the previous paper by the authors [3] some characteristic phenomena were considered which arise in the usual strong interaction theory in the limit of massless pion. It appears that since the thresholds for production of particles coincide with the boundary of the physical region, $t = 0$, the Regge trajectories and residues have singularity at $t = 0$ and also. there is a condensation of the Regge poles along the imaginary axis $\mathrm{Re}j = 0$ in complex j-plane when $t \to 0$. Both of these phenomena are also present in the discussed theory of weak interactions.

In the second section of the paper the processes $e^+ + e^- \rightleftharpoons \mu^+ + \mu^-$ are discussed which differ from elastic scattering by the "exotic" character of the quantum numbers in the t-channel. It is shown how the condensation of the poles mentioned above appears and the contribution of this condensation to the amplitude is calculated explicitly. If there exist some other singularities which are located to the right of the line $\mathrm{Re}j = 0$ the calculated asymptotic behaviour represents a lower boundary for the amplitude of the process $e^+ + e^- \to \mu^+ + \mu^-$.

In the following section we find the contribution to the amplitudes of the same process ($e^+ + e^- \to \mu^+ + \mu^-$), arising from the Regge-cut connected with the reggezation of two leptons in the t-channel intermediate state. The contribution of the cut is asymptotically of the same order of magnitude as is the contribution of the above-mentioned condensation. Because the considered particles are massless, the character of the Regge branch point turns out to be rather strong. This leads to the unusual situation when the contribution to the amplitude contains no free parameters except the mass units (of order of $\sim G^{-\frac{1}{2}}$) in which s and t are measured, these units being present only logarithmically.

In sect. 4 of the paper elastic lepton scattering is considered. The main difference from the process $e^+ + e^- \to \mu^+ + \mu^-$ is that there must exist some singularity which is located to the right of the line $\mathrm{Re}j = 0$. This provides the positivity of imaginary parts of the partial waves in the direct s-channel. In the frame of our approach it can be seen that at $t = 0$ the position of this singularity $\alpha(0) \leqslant 1$. The arguments are much the same as those presented recently in ref. [4]. Thus, for the total lepton-lepton cross section we get

$$\mathrm{const} \cdot s^{-1} < \sigma_{\mathrm{tot}} < \mathrm{const} \cdot s^{\epsilon} ,$$

where ϵ is an arbitrary small number. Whereas the upper limit has been already discussed (see for example ref. [5]) it seems to us that the lower boundary is obtained for the first time.

In the conclusion to the paper we discuss the experimental possibilities of the verification of the theory. It is estimated that in order to have the sufficiently high energies one should have colliding lepton beams with energy $(2-2.5) \times 10^3$ GeV of each beam. It is necessary then to measure the differential cross section $d\sigma/d\Omega \gtrsim 1$ pb/sr for the process $e^+ + e^- \to \mu^+ + \mu^-$ (or, maybe, several orders larger) and the total lepton-lepton cross section which magnitude might be of an order of ~ 5 nb. The last figure is obtained if one assumes that there exists the "Pomeranchuk trajectory" passing through $j = 1$ at $t = 0$. In this latter case $\sigma_{\mathrm{tot}} \sim G$, since G is the only dimensional constant which is present in the theory.

2. $e^+ + e^- \to \mu^+ + \mu^-$ production at high energies

We shall start the discussion by consideration of the process $e^+ + e^- \to \mu^+ + \mu^-$ (or the inverse one). In comparison with elastic lepton-lepton (or lepton-antilepton) scattering there are some simplifications since the quantum numbers of the t-channel coincide here with the quantum numbers of the system $\nu_e \bar{\nu}_\mu$ (or $\bar{\nu}_e \nu_\mu$). Therefore, this channel can be considered as an "exotic" one (since it has electron and muonic charges not equal to zero). At the same time the reaction $e^+ + e^- \to \mu^+ + \mu^-$ is easily detected in colliding beams experiments.

There is a set of possible intermediate states in the t-channel unitarity condition: $\nu_e \bar{\nu}_\mu$, $e^- \mu^+$ and many-particle states. Let us consider $Gs \gg 1$ and $G|t| \ll 1$. If, in spite of the condition $G|t| \ll 1$, $|t| \gg (m_\mu + m_e)^2$, we can put $m_\mu = m_e = 0$. By this we actually suppose that all the amplitudes of the leptonic proceses do not go to infinity when $m_\mu \to 0$, $m_e \to 0$, so that in the massless limit there exists some reasonable theory in which the long-range forces are connected with the exchange of e and μ pairs as well as with neutrino pairs. The analogous question has been discussed in detail for the case of the usual strong interactions in connection with the vanishing pion mass limit [3].

From the dimension reasons it is obvious that at $m_\mu = m_e = 0$ the many-particle contributions to the unitarity condition contain some extra powers of Gt in comparison with the two-particle contribution. The extra t powers are just the phase volume while G comes in because at $m_\mu = m_e = 0$ it appears to be the only dimension constant. In the physical region of the t-channel it is then clear that provided $Gt \ll 1$ the many-particle contributions are small in comparison with the two-particle contribution. However, the case $Gs \gg 1$, $Gt \ll 1$ requires the analytical continuation of the t-channel unitarity condition in s. It is not then obvious that many-particle contributions in t-channel unitarity are altogether negligible. Moreover, as we know by experience concerning the investigation of strong interactions, the many-particle states in the t-channel can lead to the very important phenomenon generation of the Regge cuts in the complex angular momentum plane. Nevertheless the investigation of the asymptotic behaviour of the amplitudes with "pure" Regge poles has proved to be very useful. Keeping this in mind we shall attempt to consider weak processes at high energies using mainly the two-particle unitarity in the t-channel but taking into account possible appearance of Regge cuts in the complex angular momentum plane.

As far as hadrons in the t-channel intermediate states are concerned, we can note that in the case of the $e^+ + e^- \to \mu^+ + \mu^-$ reaction hadron pairs can only be produced together with lepton pairs $\nu_e \bar{\nu}_\mu$ or $e^- \mu^+$. Therefore we neglect their contribution for the same reasons as with the contribution of many-particle "pure" leptonic states. For elastic lepton scattering a single hadron pair can appear in the t-channel. We shall discuss this possibility in the next sections.

Since we are going to investigate the $e^+ + e^- \to \mu^+ + \mu^-$ process by means of the t-channel unitarity condition in which $\nu_e \bar{\nu}_\mu$ and $e^- \mu^+$ intermediate states are taken into account, we must consider in the s-channel three coupled amplitudes (fig. 1): $\nu_e + \bar{\nu}_e \to \nu_\mu + \bar{\nu}_\mu$ (the amplitude $F(s,t)$), $\nu_e + e^+ \to \nu_\mu + \mu^+$ (the amplitude $G(s,t)$) and $e^- + e^+ \to \mu^- + \mu^+$ (the amplitude $H(s,t)$). The t-channel processes will be $\nu_e + \bar{\nu}_\mu \to \nu_e + \bar{\nu}_\mu$, $\nu_e + \bar{\nu}_\mu \to e^- + \mu^+$ and $e^- + \mu^+ \to e^- + \mu^+$ respectively.

Let us write down explicitely some general formulae useful for the future. Since the helicities of the initial and the final states in the t-channel are equal to unity, the partial waves with integer values of the momentum $j = n$ are defined by

$$f_n(t) = \int_{-1}^{+1} \tfrac{1}{2}\, \mathrm{d}z\, F(s,t) d^n_{11}(z)\,, \tag{1}$$

where $s = -\frac{1}{2} t(1-z)$ and $d^n_{11}(z)$ are the usual d-functions (see for example ref. [6]). Analogously to (1) the partial waves $g_n(t)$ and $h_n(t)$ can be defined in terms of the amplitudes $G(s,t)$ and $H(s,t)$.

In order to continue $f_n(t)$ to complex values of the angular momentum j we must assume, as usual, the validity of the dispersion relations in the s-variable (the sufficient number of substractions, though not written down, are implied):

$$F(s,t) = \frac{1}{\pi} \int_0^\infty \frac{F_1(s',t)\,\mathrm{d}s'}{s'-s} + \frac{1}{\pi} \int_0^\infty \frac{F_2(u',t)\,\mathrm{d}u'}{u'-u}\,, \quad s+t+u=0. \tag{2}$$

Substituting (2) into (1) and using the equations

$$\int_{-1}^{+1} \frac{d^n_{11}(z)}{z'-z} \frac{\mathrm{d}z}{2} = Q^n_{11}(z'), \quad \int_{-1}^{+1} \frac{d^n_{11}(z)}{z'+z} \frac{\mathrm{d}z}{2} = -Q^n_{11}(-z') = (-1)^n \frac{Q^n_{1-1}(z')}{n(n+1)}\,, \tag{3}$$

we get for the partial waves with positive and negative signatures

$$f^\pm_j(t) = \frac{1}{\pi} \int_1^\infty \mathrm{d}z F_1(z,t)\, Q^j_{11}(z) \pm \frac{1}{\pi} \frac{1}{j(j+1)} \int_1^\infty \mathrm{d}z F_2(z_1 t) Q^j_{1-1}(z)\,. \tag{4}$$

This equation gives the analytical continuation of the partial waves which decrease in the right half of the j-plane.

The amplitude $F(s,t)$ can be obtained now by Sommerfield-Watson representation:

$$F(s,t) = F^+(s,t) + F^-(s,t),$$

$$F^\pm(s,t) = \tfrac{1}{4} i \int_{C-i\infty}^{C+i\infty} \mathrm{d}j\, \frac{2j+1}{\sin \pi j} f^\pm_j(t)\, [-d^j_{1-1}(-z) \pm d^j_{11}(z)]\,, \tag{5}$$

which gives at, $s \to \infty$,

$$F^{\pm}(s,t)=\tfrac{1}{4}i\int_{C-i\infty}^{C+i\infty} \mathrm{d}j\ \frac{(2j+1)j(j+1)\Gamma(2j+1)}{\sin\pi j\,[\Gamma(j+2)]^2} f_j^{\pm}(t)\left[\left(-\frac{s}{t}\right)^j \pm\left(\frac{s}{t}\right)^j\right]. \tag{6}$$

At last, the expression for the absortive part $F_1^{\pm}(s,t)$ is

$$F_1^{\pm}(s,t)=-\tfrac{1}{4}i\int_{C-i\infty}^{C+i\infty} \mathrm{d}j\ \frac{(2j+1)j(j+1)\Gamma(2j+1)}{[\Gamma(j+2)]^2} f_j^{\pm}(t)\left(\frac{s}{t}\right)^j . \tag{7}$$

The equations analogous to (1)–(7) can also be written for the amplitudes $G(s,t)$ and $H(s,t)$ and the partial amplitudes $g_j(t)$ and $h_j(t)$.

The unitarity conditions for f_j, g_j and h_j have the form (see fig. 1):

$$\frac{1}{2i}(f_j-f_j^{\mathrm{II}})=f_jf_j^{\mathrm{II}}+g_jg_j^{\mathrm{II}},\quad \frac{1}{2i}(g_j-g_j^{\mathrm{II}})=f_jg_j^{\mathrm{II}}+g_jh_j^{\mathrm{II}},\quad \frac{1}{2i}(h_j-h_j^{\mathrm{II}})=g_jg_j^{\mathrm{II}}+h_jh_j^{\mathrm{II}}. \tag{8}$$

Here $f_j^{\mathrm{II}}(t)=f_{j^*}^*$, $g_j^{\mathrm{II}}=g_{j^*}^*$, ... are the values of the amplitudes $f_j(t)$,$g_j(t)$ ... on the second sheet of the t-plane. Eqs. (8) are correct separately for the amplitudes for positive and negative signature. Each amplitude f_j, f_j^{II}, g_j, etc., should have, of course, the sign indicating signature, e.g. $f_j^{+\mathrm{II}}$. We omit it for the sake of abbreviation.

If $\nu_e\nu_\mu \rightleftharpoons e\mu$ symmetry holds on at high energies (masses are neglected) then, obviously, $F(s,t)=H(s,t)$ and $f_j(t)=h_j(t)$. Thus, we obtain from (8):

$$\begin{aligned}&\frac{1}{2i}(a_j-a_j^{\mathrm{II}})=a_ja_j^{\mathrm{II}}, \qquad \frac{1}{2i}(b_j-b_j^{\mathrm{II}})=b_jb_j^{\mathrm{II}},\\ &a_j=f_j+g_j, \qquad b_j=f_j-g_j, \qquad h_j=f_j,\end{aligned} \tag{9}$$

so that the unitarity conditions are diagonalized for $f_j \pm g_j$. Since $f_j=h_j=\frac{1}{2}(a_j+b_j)$ and $g_j=\frac{1}{2}(a_j-b_j)$ and taking into acount the fact that there exist separately $a_j^{\pm}$ and $b_j^{\pm}$ with positive and negative signatures, we can represent the amplitudes F, G and H in the form

$$\begin{aligned}&F_{\nu_e\bar\nu_e\to\nu_\mu\bar\nu_\mu}(s,t)=H_{e^-e^+\to\mu^-\mu^+}(s,t)=\tfrac{1}{2}[A^+(s,t)+A^-(s,t)+B^+(s,t)+B^-(s,t)],\\ &G_{\nu_e e^+\to\nu_\mu\mu^+}(s,t)=\tfrac{1}{2}[A^+(s,t)+A^-(s,t)-B^+(s,t)-B^-(s,t)].\end{aligned} \tag{10}$$

The amplitudes $A^{\pm}$ and $B^{\pm}$ are expressed in terms of the partial waves $a_j^{\pm}$ and $b_j^{\pm}$ by eqs. (5) and (6).

As is well known, at non-integer values of the momentum j the partial waves are real below the two-particle threshold only after the extraction of the factor t^j (or $(t-4\mu^2)^j$ if a mass $\mu \neq 0$).

Introducing $\alpha_j(t)=a_j(t)/t^j$ we have for $\alpha_j(t)$

$$\frac{1}{2i}(\alpha_j-\alpha_j^{\mathrm{II}})=t^j\alpha_j\alpha_j^{\mathrm{II}} . \tag{11}$$

This equation is, of course, valid for both signatures $\alpha_j^{\pm}$ as well as for the functions $\beta_j^{\pm} = b_j^{\pm}/t^j$. As usually eq. (11) can be easily solved by consideration of the inverse amplitude $1/\alpha_j$ since

$$\frac{1}{2i}\left(\frac{1}{\alpha_j} - \left(\frac{1}{\alpha_j}\right)^{\mathrm{II}}\right) = -t^j .$$

Thus we get (see ref. [3])

$$\alpha_j(t) = \frac{1}{\Lambda_j(t) + \chi_j(t)} , \tag{12}$$

where $\Lambda_j(t)$ is a function which has no singularity as $t = 0$ and

$$\chi_j(t) = \frac{1}{\sin \pi j} \left[(-t)^j - \Sigma_n (-t)^n\right] . \tag{13}$$

The first term in eq. (13) provides the validity of the unitarity condition (11) for α_j. The $\Sigma_n \ldots$, terms can be understood as some substractions which can also be included in the regular function $\Lambda_j(t)$. The meaning of these terms is as follows. At $j = n$, where n is integer, $(-t)/\sin \pi j = \infty$ so that the amplitude α_j would tend to zero if these substractive terms were absent. We do not see any reasons for the vanishing of α_j at integer $j = n$. If, on the contrary, these terms are included, the equation

$$\frac{1}{2i}\left[\alpha_j^{-1} - (\alpha_j^{\mathrm{II}})^{-1}\right] = -t^j$$

remains valid for integer j. One can also check the presence of such terms by means of some potential model using the results of ref. [7]. We can note, at last, that the absence of the substractions discussed changes very slightly the final results.

Eq. (12) is the solution of the two-particle unitarity condition. Therefore the singularities of α_j are the Regge poles corresponding to vanishing of the denominator in (12). Expanding $\Lambda_j(t)$ near some point $j = \alpha_0$ and $t = 0$ as $\Lambda_j(t) = [j - \alpha_0 - ct]/b$, we see that the trajectories of these poles have a singularity at $t = 0$ whose character depends on the values of the intercept α_0. For $\alpha_0 > 0$

$$\begin{aligned} \alpha(t) &= \alpha_0 - b\,(-t)^{\alpha_0}/\sin \pi\, \alpha_0 + Ct, & \alpha_0 &\neq n , \\ \alpha(t) &= n - \frac{b}{\pi} t^n \ln(-t) + Ct, & \alpha_0 &= n . \end{aligned} \tag{14}$$

This threshold behaviour of the trajectories had been previously discussed in ref. [8] and recently for the massless case in a paper of the authors [3].

The denominator of eq. (12) is a rapidly varying function of t and j at small $t \to 0$ and $j \to 0$ because of the singular character of the function $\chi_j(t)$ in this region. This leads to the appearance of an infinite number of poles which move to the point $j = 0$ as $t \to 0$. The analogous threshold condensation of the poles and its contribution to the asymptotics has been considered before in ref. [9] in the case of non-vanishing masses. In this latter case the condensation removes to the point $j = -\frac{1}{2}$ as $t \to 4\mu^2$. The most interesting feature of the massless case is that the point $t = 0$ is now the

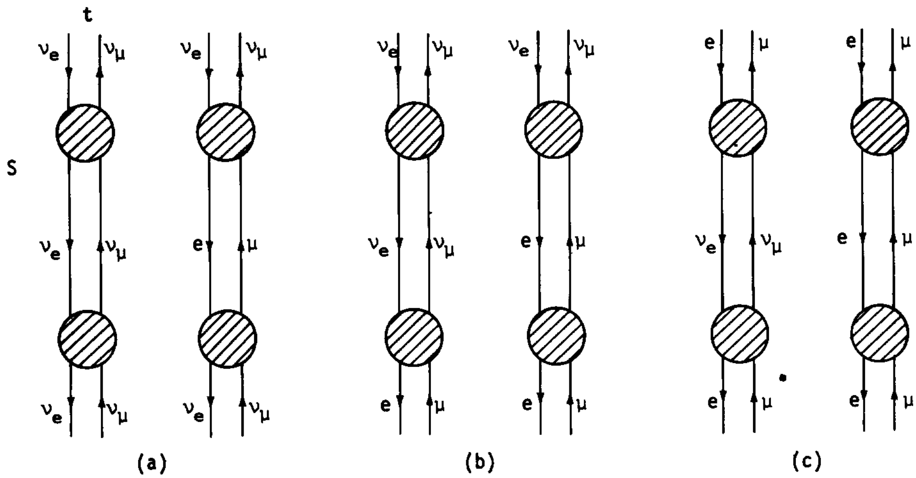

Fig. 1. Illustration for t-channel two particle unitarity condition.

physical one, and, besides, that the position of the condensation at $j = 0$ provides larger asymptotics than that at $j = -\frac{1}{2}$ (when $\mu \neq 0$).

Since at present we consider the amplitudes with the "exotic" quantum numbers of the t-channel, we assume that there are no Regge poles with positive intercept $\alpha_0 > 0$. If this hypothesis is incorrect then the asymptotic expressions for F, G and H, obtained below, should be understood as the lower bound for the amplitudes.

Let us represent the function $\Lambda_j(t)$ near $j = 0$ and $t = 0$ as

$$\Lambda_j(t) = a + bj\,, \tag{15}$$

where a and b are some constants. We neglect the term proportional to t since, as it will be seen below, the essential values of j will be of order

$$j \sim \frac{1}{\ln(-1/t)} \gg t$$

as $t \to 0$. We can consider the constants a and b as some dimensionless quantities if t is measured in the natural units G^{-1}, the coupling constant G being the only dimensional constant in the theory. Then by the statement "t is small" we mean actually $Gt \ll 1$, $j \gg t$ means $j \gg Gt$, and so on.

We get from (12), (13) and (15) at $j \simeq 0$ and $t \simeq 0$

$$\alpha_j = \frac{\pi j}{e^{-\tau j} - 1 + \pi j\,[a + bj]}\,, \quad \tau = \ln(-1/t) > 0. \tag{16}$$

The vanishing of the denominator gives us the trajectories of the poles:

$$j_n(t) = \frac{2\pi i n}{\tau} - \frac{4\pi^2 n^2 \gamma}{\tau^3}\,, \quad \gamma = \tfrac{1}{2}\pi^2 a^2 + \pi b\,, \quad n = \pm 1, \pm 2, \ldots .$$

The linear in j term (πaj) is excluded by redefinition of the units in which t is measured. In order of magnitude these units remain $\sim G^{-1}$: $\tau = \ln(-1/Gt)$.

Substituting (16) in (6), we obtain the asymptotics of the amplitudes with positive and negative signatures as the sums over the contributions of the poles (we would remind that α_j differs from a_j by the factor t^j which cancels out with t^{-j} in eq. (6)):

$$A^+(s,t) = -\frac{4\pi^2}{\tau^2}\sum_{n=1}^{\infty}\exp\left[-\frac{4\pi^2n^2}{\tau^3}\gamma^+\xi\right]n\sin\frac{2\pi n\xi}{\tau}$$

$$+\, i\,\frac{4\pi^4}{\tau^3}\sum_{n=i}^{\infty}\exp\left[-\frac{4\pi^2n^2}{\tau^3}\gamma^+\xi\right]n^2\cos\frac{2\pi n\xi}{\tau},$$

$$A^-(s,t) = -\frac{4\pi^6}{\tau^4}\sum_{n=1}^{\infty}\exp\left[-\frac{4\pi^2n^2}{\tau^3}\gamma^-\xi\right]n^3\sin\frac{2\pi n\xi}{\tau}$$

$$+\, i\,\frac{4\pi^4}{\tau^3}\sum_{n=1}^{\infty}\exp\left[-\frac{4\pi^2n^2}{\tau^3}\gamma^-\xi\right]n^2\cos\frac{2\pi n\xi}{\tau}\,. \tag{17}$$

The parameters γ^+ and γ^- for the positive and negative signatures are, of course, different. The amplitude of the process $e^- + e^+ \to \mu^- + \mu^+$, $H(s,t)$, is given now by formula (10) with $B^\pm(s,t)$ being determined by the same equations as (17) but with some other values of constants $\gamma^\pm$.

If $4\pi^2\xi\gamma^\pm/\tau^3 \gg 1$ the only term with $n = 1$ is essential in the sums in (17). Among the four functions A^+, A^-, B^+, B^- one should keep then only that function which correspond to the smallest value of γ. Thus, at $4\pi^2\xi\gamma/\tau^3 \gg 1$ we have

$$H_{e^-e^+\to\mu^-\mu^+}(s,t) \simeq \frac{2\pi^4}{\tau^3}\left[i\cos\frac{2\pi\xi}{\tau} - \left(\frac{\tau}{\pi^2}\right)^{\pm 1}\sin\frac{2\pi\xi}{\tau}\right]\exp\left[-\frac{4\pi^2\gamma\xi}{\tau^3}\right]. \tag{18}$$

The sign ($\pm$ 1) in the second term must be chosen in correspondence to the signature of the function remaining in the expression (10) for $H(s,t)$ (i.e. the function which correspond to the smallest value of γ in its t-channel partial wave expansion).

If, on the contrary, $4\pi^2\gamma\xi/\tau^3 \ll 1$ then the large number of terms in eqs. (17) are essential (this is, of course, purely from theoretical point of view, since the numerical convergence of the sum is perfect). This limit will be of some special interest for us in the case of elastic scattering and will be considered in details below. Here we note only that in this region $A^\pm$ functions do not vanish only in the neighbourhood of the integer values of $\xi/\tau = 1, 2, \ldots$ The heights of the peaks are proportional to $\sim \gamma^{-3/2}$ while their widths are $\sim \gamma^{1/2}$. Therefore the presence of four terms in eq. (10) leads to the result that on the pattern consisting of the main peaks some additional broader peaks with smaller heights are superimposed.

3. The contribution of the Regge cuts to asymptotics

It is well known that in relativistic theory, because of the presence of the third spectral function $\rho(s, u)$, the discontinuities of the partial waves on the left-hand cut in the t-plane have the poles at integer negative values of the angular momentum [10]. Since in the present case we consider in the intermediate state two particles with spin $\frac{1}{2}$, there must exist a pole of the discontinuity of $a_j(t)$ at $j = 0$ ("Azimov's displacement" [11]). One can easily see this by calculating the discontinuity of $\alpha_j(t) = a_j(t)/t^j$ connected with the third spectral function.

We have from (4)

$$\Delta_L \alpha_j^{\pm}(t) = \frac{1}{\pi} \int_0^{-t} \frac{ds}{t^{j+1}} \rho(s, u)\, Q_{11}^j(z) \pm \frac{1}{\pi} \frac{1}{j(j+1)} \int_0^{-t} \frac{du}{t^{j+1}} \rho(s, u) Q_{1-1}^j(z)\,, \tag{19}$$

where in the first term $z = 1 + 2s/t$ while in the second $-z = 1 + 2u/t$. At $j \to 0$

$$Q_{11}^j(z) \sim \frac{1}{j}\left(\frac{1}{z+1}\right), \quad Q_{1-1}^j(z) \sim \frac{1}{z-1}\,,$$

hence

$$\Delta_L \alpha_j^{\pm}(t) \simeq \frac{1}{2\pi j}\left[\int_0^{-t} \frac{ds}{s+t} \rho(s, u) \pm \int_0^{-t} \frac{du}{u} \rho(s, u)\right]. \tag{20}$$

Introducing $u = -s - t$ as a variable of integration in the first integral, we see that the pole is absent in the positive signature whereas for negative signature the pole term is equal to

$$\Delta_L \alpha_j^{-}(t) = -\frac{1}{\pi j} \int_0^{-t} \frac{du}{u} \rho(s, u)\,. \tag{21}$$

Note, that because of helicity structure of the theory $\rho(s, u) \sim u$ at small u (as well as the whole amplitude) and so there is no divergency at $u \to 0$. The absence of the pole in the positive signature amplitude is a manifestation of the general rule about the absence of the poles in the points of "right" signature.

The existence of the pole in $\Delta_L \alpha_j^-$ leads, as is well-known, to two possibilities The first is that there is an essential singularity of α_j^- at $j = 0$ [10]. The second possibility is that at $j \to 0$ the many-particle contribution in the unitarity condition cancel out the pole of the second order, appearing in the right-hand side of the two-particle unitarity condition (11). Then both sides have the pole of only the first order and unitarity is not violated. In this case the moving cuts appear which are the result of the reggeization of particles in the intermediate state. Despite the reggeization of leptons seeming rather extravagant at the present time, it appears to us to be a natural possibility. Indeed, from the point of view, developing in this paper, in the region $Gs > 1$ weak interactions do not differ, in principle, from the strong interactions. In comparison to the strong interactions it is only the scale of masses which

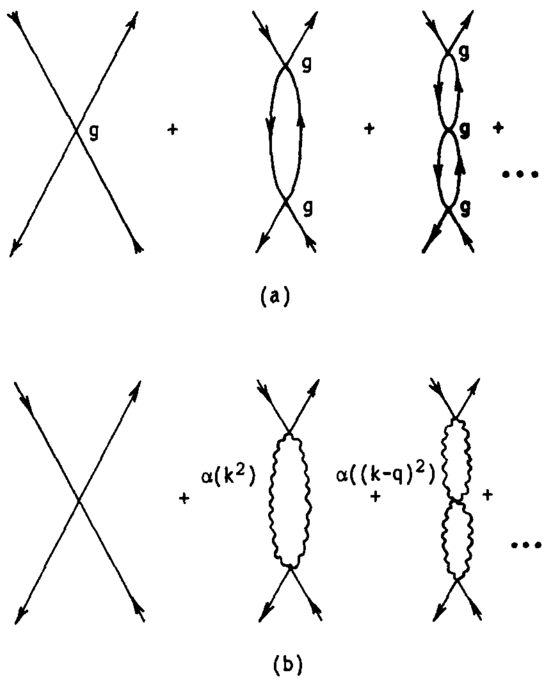

Fig. 2. (a) Chain of diagrams corresponding to the partial wave of eq. (12) (b) Chain of reggeon diagrams corresponding to reggeization of particles in the intermediate state.

change: the characteristic mass $G^{-1/2}$ is much larger than the characteristic hadron masses. This reflects in the fact that after the reggeization the slopes of lepton trajectories must be in order of magnitude *

$$\alpha(t) = 1/2 + \alpha' t\,, \qquad \alpha' \simeq G\,. \tag{22}$$

Thus, the heavier particles of the Regge occurence (22) must have masses of order of hundreds or even thousands GeV.

Let us consider the contribution of the two-reggeon cut in the negative-signature partial wave α_j^-. As it was mentioned above, this cut provides the validity of the unitarity condition for α_j^- and we will not describe the details of this mechanism, because the situation is completely the same as that considered by Mandelstam [12]. The simple meaning of eqs. (12) and (13) for α_j is that they correspond to the summation of t-channel loop diagrams drawn in fig. 2a. The function $\chi_j(t)$ is the lepton loop, while $\Lambda_j(t) = [g_j(t)]^{-1}$ where $g_j(t)$ is the amplitude of the transition $\ell+\bar{\ell} \to \ell+\bar{\ell}$

* We do not consider the $(Gt)^{1/2}$ term in the trajectory, possible in the fermion case. Its presence does not influence much what follows.

shown in fig. 2. (We consider somewhat "conditioned" leptons arising after diagonalization of the unitarity condition (8) as it was demonstrated by eq. (9)). The reggeization of the leptons preserves the form (12) for α_j^- since it is now determined by the set of the reggeon diagram of fig. 2b. The wavy lines represent the reggeons. There is, of course, a question whether the vertex $\ell + \bar{\ell} \to 2$ reggeons coincides with 2 reggeons $\to$ 2 reggeons vertex. However, as it will be seen below, after the extraction of some rapidly varying factors, we take a limit in which the reggeon trajectories $\alpha_1 = \alpha_2 = 1/2$. In this limit both vertices are certainly the same, which proves the validity of the form (12). The difference with the previous case consists in the explicit form of the function $\chi_j(t)$ which can now be calculated by means of reggeon diagram techniques [13].

It is convenient to introduce the function

$$\alpha_j^-(t) = j\,\alpha_j^-\,(t) \tag{23}$$

which enters directly into the integral determining the asymptotic behaviour of $A_1(s,t)$ at small j (see eq. (7)):

$$A_1^-(s,t) = -\tfrac{1}{4}\,i \int_{C-i\infty}^{C+i\infty} \mathrm{d}j\,\bar{\alpha}_j(t)\,s^j\,. \tag{24}$$

As it was mentioned above the general form of $\bar{\alpha}_j$ is

$$\bar{\alpha}_j = \frac{1}{\bar{\Lambda}_j(t) + \bar{\chi}_j(j)} \tag{25}$$

where $\bar{\Lambda}_j(t)$ has no singularity at $t = 0$, and the function $\bar{\chi}_j(t)$ can be found by the rules formulated in ref. [13] for calculation of the reggeon loops. It is convenient to start the consideration in the region of $-t = q^2 > 0$ and then to do the analitical continuation to positive values of t. The expression for $\bar{\chi}_j$ at small t and j can be written as follows:

$$\bar{\chi}_j(q^2) = C \int \frac{\mathrm{d}^2k}{j - \alpha(k^2) - \alpha((q-k)^2) + 1}\,\frac{k\cdot(q-k)}{k^2(q-k)^2}\,p^{2[j-\alpha(k^2)-\alpha((q-k)^2)+1]} \tag{26}$$

The equation (26) requires some clarification. The constant C is the normalization factor. The two-dimensional integration (d^2k) and the denominator $j - \alpha(k^2) - \alpha((q-k)^2) + 1$ correspond to the usual rules of the reggeon diagram techniques [13]. The denominators $1/k^2$ and $1/(q-k)^2$ arise from the signature factors of the reggeons $1/\sin \pi(\alpha(k^2) - 1/2)$ and $1/\sin \pi(\alpha((q-k)^2) - 1/2)$ which tend to infinity at $k^2 = 0$ and $(q-k)^2 = 0$. In the region of positive t the contribution of these poles provides the cancelation of the pole at $j = 0$ in two- and four-particle contributions in unitarity condition for $\bar{\alpha}_j$ (for more details see ref. [12]). The factor $k\cdot(q-k)$ is connected with the fermion character of the reggeons. At last, the factor $p^{2[j-\dots]}$ reflects the threshold behaviour of the amplitudes of reggeon's production, p being the momentum of the relative motion of reggeons. This latter factor is usually neglected when one is interested only in the form of $\bar{\chi}_j$ in the immediate neighbourhood of moving singularities arising from vanishing of denominator $j-\alpha(k^2)-\alpha((q-k)^2)+1=0$. The appearance of this factor can be traced following the results of ref. [14].

For the case of non-reggeized leptons, $\alpha(k^2)=\alpha(q-k)^2)=1/2$ the integral (26) must give an expression for $\bar{\chi}_j(t)$ which agrees with formula (13) for $\chi_j(t)$ for small j and t, i.e. $\bar{\chi}_j = 1/\pi j^2[(-t)^j-1]$ $(\bar{\chi}_j=\chi_j/j)$. This will allow us to determine the numerical constant C.

Indeed, using the value of p^2

$$\begin{aligned} p^2 &= -\frac{1}{4q^2}\,[q^4 - 2q^2(k^2+(q-k)^2) + [k^2-(q-k)^2]^2] \\ &= k^2\,(1-\cos^2\theta), \qquad \cos\theta = q\cdot k/|q|\,|k|\,, \end{aligned} \tag{27}$$

we can rewrite (26) in the case $\alpha(k^2)=\alpha((q-k)^2)=1/2$ and $j\to 0$ in the form:

$$\bar{\chi}_j = \frac{C}{j}\int \mathrm{d}^2k\,\frac{k\cdot(q-k)}{k^2(q-k)^2}\,k^{2j}\,. \tag{28}$$

At $j=0$ the logarithmic divergency of the integral at small k^2 is cut off by the value of q^2. Therefore for calculation of the singular part of (28) the values $k^2 \gg q^2$ are essential, so that we get:

$$\bar{\chi}_j = \frac{C}{j}\int_{q^2}^{\Lambda^2}\frac{\pi\,\mathrm{d}(k^2)}{k^2}\,k^{2j} = -\frac{\pi C}{j^2}\,[\Lambda^{2j}-q^{2j}] = \frac{\pi C}{j^2}\,[(-t)^j-1]\,. \tag{29}$$

(The region $k^2\sim 1$ does not give any singular contribution. So we cut off the integral at large k^2 at $k^2=\Lambda^2$. As $j\to 0$, $\Lambda^{2j}\to 1$). The expression (29) does agree with (13) $(\bar{\chi}_j = 1/\pi j^2\,[(-t)^j-1])$ if the numerical value of C is $C=1/\pi^2$.

We are interested in the function $\bar{\chi}_j$ in two quite different regions of $q^2=-t$ and j. In the first region $j\sim(\ln 1/q^2)^{-1}\gg q^2 (q^2\to 0)$. It is easy to see that here the expression $j-\alpha(k^2)-\alpha((k-q)^2)+1$ can be safely replaced by j, and we again arrive at (29). Thus the reggeization of leptons in the intermediate state do not influence on the form of the partial wave in the region where the condensation of the poles, considered above, is placed.

On the other hand at $j\sim t$ there are some new singularities of $\bar{\chi}_j$ namely moving branch points, which contribution to the asymptotics should be added to the contribution of the poles to $A^-(s,t)$ (eq. (17)). At $j\sim q^2$ the factor $p^{2[j\cdots]}$ can be neglected and we have

$$\bar{\chi}_j(q^2) = \frac{1}{\pi^2}\int\frac{\mathrm{d}^2k}{j-\alpha(k^2)-\alpha(q-k)^2)+1}\;\frac{k\cdot(q-k)}{k^2(q-k)^2}\,. \tag{30}$$

Putting $\alpha(k^2) = 1/2 + \alpha' k^2$ we obtain by straightforward calculation

$$\bar{\chi}_j(q^2) = \frac{1}{\pi j} \ln \frac{(2j/\alpha' q^2) + 1}{((j/\alpha' q^2) + 1)^2} . \tag{31}$$

The singularity $j = -\frac{1}{2}\alpha' q^2$ is the usual Mandelstam branch point enhanced by the factor $1/j$ at small j due to the masslessness of the particles involved. The branch point $j = -\alpha q^2$ is characteristic for the massless case. Since at small j (and $q^2 \sim j$)$\bar{\chi}_j \gg \bar{\Lambda}_j$ the partial wave $\bar{\alpha}_j$ is determined in this region uniquely:

$$\bar{\alpha}_j = \frac{\pi j}{\ln \dfrac{(2j/\alpha' q^2) + 1}{((j/\alpha' q^2) + 1)^2}} . \tag{32}$$

Substituting (32) into (24) it is easy to get the contribution of the cuts to the asymptotics of $A^-(s, t)$ which should be added to (17). For $\alpha' q^2 \xi \gg 1$:

$$\delta A^-(s, t) = -\tfrac{1}{4}\pi \frac{\alpha' q^2}{\xi} e^{-\frac{1}{2}\alpha' q^2 \xi} [i - \tfrac{1}{4}\pi\alpha' q^2] \frac{1}{\ln^2 (\frac{1}{8}\alpha' q^2 \xi) + \pi^2} . \tag{33}$$

The region of the essential variation of expression (17) is $\tau \sim \xi$ or $\tau^3 \sim \xi$. In this region $q^2 \xi \ll 1$ so that the contribution of the cut can be written in the simplest form:

$$\delta A^-(s, t) = \frac{\pi^2}{2\xi^2 \tau^2} \left[i - \left(\frac{\pi}{\xi}\right) \right]. \tag{34}$$

It is interesting that this value does not depend on any unknown parameters (except of the units in which s and t are measured, entering the logarithm in (34)).

If we return to the physical amplitudes F, G, and H (eq. (10)) we see that the contribution (34) should be added to the amplitude of the process $e^+ + e^- \to \mu^+ + \mu^-$ (since it contains both functions $(+A^-)$ and $(+B^-)$) whereas in the amplitude of the transition $\nu_e + e^+ \to \nu_\mu + \mu^+$ this contribution is cancelled out.

4. Elastic lepton scattering

As it was already mentioned the essential difference between the process $e^- + e^+ \to \mu^- + \mu^+$ and the elastic scattering consists in the fact that in the latter case the t-channel has vacuum quantum numbers. This leads to some complications which are absent for the reaction $e^- + e^+ \to \mu^- + \mu^+$.

(i) In the case of elastic scattering of charged leptons (electrons, muons) the photon exchange in t-channel is possible and because of this the electromagnetic interaction gives here much more essential contribution than in the case of $e^- + e^+ \to \mu^- + \mu^+$, where only the s-channel photon is present. The above considered condensation of the poles as well as the cut contribution gives in order of magnitude for the cross-section of the process $e^- + e^+ \to \mu^- + \mu^+$ at small angles $d\sigma/d\Omega \sim 1/s$. The electromagnetic cross-section at zero angle is equal here to $\alpha^2/2s$, $\alpha = 1/137$, i.e. four orders less than the "weak" cross-section. In the case of elastic scattering the electromagnetic cross section is $(\alpha^2/t^2)s$ so that the ratio $(d\sigma/d\Omega)_{el}/(d\sigma/d\Omega)_{weak} \sim (\alpha s/t)^2 \sim \alpha^2/\theta^4$. Thus at small angles $\theta < \sqrt{\alpha} \sim 0.1$ the electromagnetic interaction is more essential than the weak one.

(ii) In the case of elastic scattering a single hadron pair can appear in t-channel. The situation is probably as follows. When hadron-antihadron contribution to t-channel unitarity condition is continued to the physical region of s-channel there enters the amplitude of the process $\ell + h \to \ell + h$(ℓ is lepton, h is hadron) at very high energy and relatively small momentum transfer t. Though we are interested in those values of t which are small in comparison with G^{-1} $(Gt \ll 1)$, thery are still very large in the scale of hadron masses. If the $\ell + h \to \ell + h$ amplitude rapidly decreases as a function of momentum transfer (which looks to be an ordinary situation for hadron processes) than the contribution of hadron pair will be very small. If, however, there exist some hadrons (partons?) for which this decrease is not so strong, these hadron pairs should be taken into account in t-channel unitarity condition together with lepton pairs. As a result one will have a system of coupled equations of the type of (8) and, after diagonalization, the elastic amplitude will be represented as a sum of terms of the same kind similarly to formulae (10) for inelastic amplitudes.

(iii) We shall prove now that expression (17) (or the sum of several terms of that type) cannot be completely correct for elastic amplitude. The reason is that the imaginary parts of the s-channel partial waves as calculated by (17) turn out to be altering in signs. In order to preserve s-channel unitarity there must exist at any rate one additional singularity (for instance, a Regge pole) placed at $t = 0$ to the right of the point $j = 0$. We shall show that such a singularity guarantees the validity of the unitarity condition provided the contribution of this singularity itself to the imaginary parts of the partial waves is positive. Thus the asymptotic behaviour of the elastic amplitude must be determined by some singularity located to the right of the point $j = 0$. The specific character of the situation is that due to the masslessness of the particles in t-channel the trajectory of this singularity is itself singular at the point $t = 0$ (eq. (14)).

As it will be seen below, inspite of the existence of massless particles, the intercept of the discussed singularity cannot be greater than unity, $\alpha(0) \leqslant 1$. The argumentation is much the same as that was given in [4]. Finally we obtain that the asymptotic behaviour of the total lepton cross section is restricted by limits:

$$\text{const} \cdot 1/s < \sigma_{tot} < \text{const} \cdot s^{\epsilon} \tag{35}$$

where ϵ is an arbitrary small fixed number.

To calculate the partial waves of the direct channel $f_\rho(s)$ (ρ is the impact parameter) is is convenient before to obtain some simple form for the sums entering eq. (17) in the case $4\pi^2\gamma\xi/\tau^3 \ll 1$. It is clear that on this condition many terms are essential in each sum in eq. (17). Let us consider for example the imaginary part $A_1(s,t)$. If not only $4\pi^2\gamma\xi/\tau^3 \ll 1$ but also $\xi/\tau \ll 1$ than the sums over n can be replaced by the integral and we easily get:

$$A_1 = \left(\frac{\pi\tau}{4\gamma\xi}\right)^{3/2} \left[1 - \frac{\tau\xi}{2\gamma}\right] e^{-\tau\xi/4\gamma} , \qquad \xi/\tau \ll 1 . \tag{36}$$

Thus at very small momentum transfer when $\tau = \ln(-1/t) \gg \xi = \ln s$ the amplitude A_1 is exponentially small.

Let it be now that $\xi/\tau \gtrsim 1$ but still $4\pi^2\gamma\xi/\tau^3 \ll 1$. Then A_1 is not equal to zero only in neighbourhood of the integer values of $\xi/\tau = m = 1, 2, 3 \ldots$. If in the neighbourhood of $\xi/\tau = m$ we write A_1 in the form:

$$A_1 = \frac{2\pi^4}{\tau^3} \sum_{n=-\infty}^{\infty} n^2 \exp\left[-\frac{4\pi\gamma\xi}{\tau^3} n^2 + 2\pi i \left(\frac{\xi}{\tau} - m\right)\right] , \tag{37}$$

then it is easy to show that provided $\xi/\tau - m \ll 1$ and $4\pi^2\gamma\xi\tau^3 \ll 1$ we can again replace the summation by integration. The integral is easily calculated and the final expression for A_1 appears to be the sum of nonoverlaping peaks near the values of $\xi/\tau \approx m$:

$$A_1 = \sum_{m=1,2,\ldots} \left(\frac{\pi\tau}{4\gamma\xi}\right)^{3/2} \left[1 - \frac{\tau^3}{2\gamma\xi}\left(\frac{\xi}{\tau} - m\right)^2\right] \exp\left[-\frac{\tau^3}{4\gamma\xi}\left(\frac{\xi}{\tau} - m\right)^2\right] . \tag{38}$$

At smaller τ (larger $|t|$), when $\gamma\xi/\tau^3 \sim 1$ the peaks begin to overlap and at $4\pi^2\gamma\xi/\tau^3 \gg 1$ A_1 has the form

$$A_1 = \frac{4\pi^4}{\tau^3} \exp\left(-\frac{4\pi^2\gamma\xi}{\tau^3}\right) \cos\frac{2\pi\xi}{\tau} , \tag{39}$$

and is small.

The expression (38) with the same accuracy can be rewritten as the distribution in τ

$$A_1 = \sum_{m=1,2,\ldots} \left(\frac{\pi}{4\gamma m}\right)^{3/2} \left[1 - \frac{m}{2\gamma}\left(\tau - \frac{\xi}{m}\right)^2\right] \exp\left[-\frac{m}{4\gamma}\left(\tau - \frac{\xi}{m}\right)^2\right] . \tag{40}$$

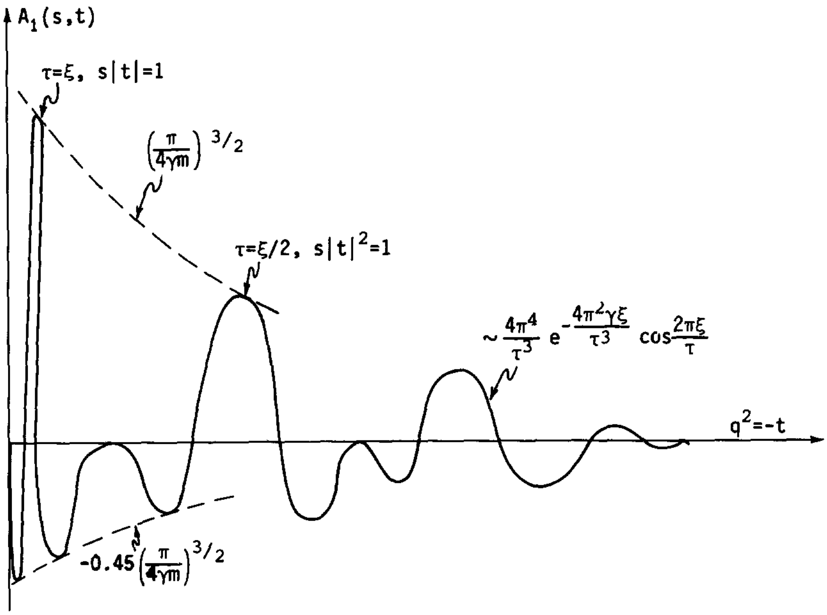

Fig. 3. The sketch of the function $A_1(s, t)$ *versus t*. The diagram is not the real plot but merely the illustration.

Because of the intricate character of t-dependence of A_1 this function is drawn quite schematically in fig. 3. Note that the height of the first peaks is of order of unity and, being so, is larger than the contribution (34) of the Regge cut, discussed in the previous section.

Starting from eq. (38) we can get the imaginary parts of s-channel partial waves. Since

$$\operatorname{Im} f_\rho(s) = \frac{1}{s} \int_0^\infty dq^2 J_0(q\rho) A_1(s, q^2) \tag{41}$$

we represent $\operatorname{Im} f_\rho$ as a sum:

$$\operatorname{Im} f_\rho = \sum_{m=1,2,..} \operatorname{Im} f_\rho^{(m)}(s),$$

$$\operatorname{Im} f_\rho^{(m)}(s) = \left(\frac{\pi}{4\gamma m}\right)^{3/2} \frac{1}{s^{1+1/m}} \int_{-\infty}^{+\infty} dz\, e^{-z-mz^2/4\gamma} \left(1 - \frac{m}{2\gamma} z^2\right) J_0\left(\rho e^{-\xi/2m-\frac{1}{2}z}\right) \tag{42}$$

We see from (42) that $\mathrm{Im} f_\rho^{(m)}(s) = s^{-1-1/m} F_m(\rho_m)$ where $\rho_m = \rho\, \mathrm{e}^{-\xi/2m} = \rho/s^{1/2m}$. Estimating the integral by the steepest descent method it is not difficult to show that at $\rho_m \gg 1$ $\mathrm{Im} f_\rho^{(m)}$ decreases with ρ_m faster than any power of $1/\rho_m$:

$$|F(\rho_m)| \sim \exp\left(-\frac{m}{\gamma} \ln^2 \rho_m\right) \tag{43}$$

Thus there exist some succession of decreasing impact parameters $\rho \sim s^{\frac{1}{2}}, s^{\frac{1}{4}}, s^{\frac{1}{6}}$, etc., corresponding to $\rho_m \sim 1$ so that at each ρ from the succession the new contributions to $\mathrm{Im} f_\rho$ arise ($\mathrm{Im} f_\rho^{(1)}, \mathrm{Im} f_\rho^{(2)}$, and so on). At the edge of each such a disc, where $\rho \sim s^{1/2m}$, $\mathrm{Im} f_\rho^{(m)} \sim 1/s^{1+1/m}$ and oscillate, while $\mathrm{Im} f_\rho^{(m')}$ with $m' > m$ $(\rho_{m'} < \rho_m)$ are small and $\mathrm{Im} f_\rho^{(m')}$ with $m' < m$ $(\rho_{m'} > \rho_m)$ does not depend on ρ and give the positive contribution of order of $\sim s^{-1-1/m}$, i.e. again small as compared with $\mathrm{Im} f_\rho^{(m)}$. In order to show that any singularity placed to the right of the point $j = 0$ provides the positiveness of the imaginary parts of partial waves we must now only demonstrate that the contribution of this singularity in $\mathrm{Im} f_\rho$ (denote it as $\mathrm{Im} \tilde{f}_\rho$) at $\rho \sim s^{1/2m}$ is larger than $s^{-1-1/m}$.

As it has been stated in the second section, the trajectory with the intercept $\alpha(0) = \alpha_0$ has at $t = 0$ singularity (14). Therefore we have for $\mathrm{Im} \tilde{f}_\rho$ in order of magnitude:

$$\mathrm{Im} \tilde{f}_\rho \sim s^{\alpha_0 - 1} \int_0^\infty \mathrm{d}q^2 J_0(q\rho) \exp\left[-b \frac{q^{2\alpha_0} \xi}{\sin \pi\alpha_0}\right], \quad 0 < \alpha_0 < 1 . \tag{44}$$

For $\xi/\rho^{2\alpha_0} \ll 1$ we can expand the exponential factor and get:

$$\mathrm{Im} \tilde{f}_\rho \sim \frac{1}{\rho^4} \left(\frac{s}{\rho^2}\right)^{\alpha_0 - 1} \tag{45}$$

In (45) the logarithmic dependence is neglected. As it is seen from its derivation the equation (45) is valid not for too large values of ρ, but only for $\rho^2 \ll s$. At $\rho^2 \sim s$ the contributions of all the singularities, lying to the left of the point α_0 are of the same order. Note, that (45) is also correct for $\alpha_0 = 1$.

At $\rho \sim s^{1/2m}$, $m = 2, 3, \ldots,$

$$\mathrm{Im} \tilde{f}_\rho \sim s^{-1-1/m} s^{(1-1/m)\alpha_0} .$$

Thus, if $\alpha_0 > 0$, $\mathrm{Im} f_\rho \gg \mathrm{Im} f_\rho^{(m)} \sim s^{-1-1/m}$ and the total imaginary part is positive indeed. For $m = 1$, $\rho^2 \sim s$, as it has been already explained, the contribution of all the singularities are of the same order and so the positiveness of the imaginary part can easily be achieved.

We see immediately from (45) that, on the other hand, $\alpha_0 \leqslant 1$. Indeed, it follows from (45) that $\mathrm{Im} f_\rho(s) > 1$ when $\rho^2 < s^{\alpha_0 - 1/\alpha_0 + 1}$. Therefore, if $\alpha_0 > 1$, $\mathrm{Im} \tilde{f}_\rho(s)$ turns out to be larger than unity (and, hence, antiunitary) inside the range ρ where the estimation (45) is correct, i.e. $\xi^{1/\alpha_0} \ll \rho^2 \ll s$. Thus we obtain the inequalities (35).

To the conclusion of this section we consider briefly lepton-antilepton backward scattering, for example $e^+ + e^- \to e^- + e^+$. The asymptotic behavior is determined here by the singularities of u-channel partial waves. Since u-channel contains two particles (not particle + antiparticle as before) the helicity here is equal to zero and the partial waves can be represented by the formula:

$$f_j^+(u) = \frac{2}{\pi} \int_0^\infty dz\, Q_j(z) A_1(s, u), \quad z = 1 + \frac{2s}{u}. \tag{46}$$

The partial waves of negative signatures are absent because of the $s \to t$ symmetry in the reaction $e^+ + e^- \to e^- + e^+$. At small values of u the absorptive part A_1 as well as the amplitude itself are proportional to u because of the conservation of helicity in backward scattering. This means that at $u \to 0$:

$$f_j^+(u) \sim u^{j+1} \tag{47}$$

which lead to the conclusion that the above considered condensation of the poles removes in u-channel to the point $j = -1$. So, if there are no singularities with intercept larger than -1, the amplitude is of order of $\sim s^{-1}$.

5. Conclusions

Let us discuss now the experimental possibilities (however poor they are) of the verification of the theory. It is necessary first of all to estimate the minimal energies which could be considered as sufficient for observation of the phenomena described. As it was already mentioned weak interactions become strong when the partial waves saturate the unitary limit. For the process $\ell + \bar{\ell} \to \nu + \bar{\nu}$ the P-wave as calculated in the first order of perturbation theory reaches its unitary limit at $s = s_0$ where s_0 is defined by

$$|f_1(s_0)| = \frac{Gs_0}{6\pi\sqrt{2}} = 1, \qquad s_0 \simeq 2.5 \cdot 10^6 \text{ GeV}^2. \tag{48}$$

We see that $f_1(s_0)$ differs from Gs by a numerical factor of order of $\sim 1/25$, which has to be taken into account [2]. Respectively, for the units of s and t it is reasonable to choose the value $s_0 = 25\, G^{-1}$ and not G^{-1} as it was supposed throughout the purely theoretical part of the paper.

Assume that the appropriate energy is of order of $s \sim 10 s_0$, i.e.

$$s \sim 25 \cdot 10^6 \text{ GeV}^2 \tag{49}$$

This corresponds to two colliding beams, each one with the energy $E = 1/2\sqrt{s} = 2500$ GeV. We will refrain from commenting on the obvious difficulties of designing such an accelerator.

On the other hand, as far as the value of the cross-section of, say, the process $e^- + e^+ \to \mu^- + \mu^+$ is concerned it does not look hopelessly small. For example, for

$s = 10s_0$, $\xi = \ln s/s_0 = 2.3$ and $(-t) = 0.1s_0$, $\tau = \ln(s_0/-t) = 2.3$ (this corresponds to the scattering angle $\theta \sim \sqrt{-t/s} \sim 0.1$ and to $-t \simeq 2.5 \cdot 10^4$ GeV^2) the contribution of the Regge cut in A^- gives $|\delta A^-|^2 \sim 0.02$ (eqs. (33)–(34)). In our normalization the cross-section is then

$$\frac{d\sigma}{d\Omega} \sim \frac{4}{s}|\delta A^-|^2 \sim 1 \text{ pb/sr}. \tag{50}$$

As to the contribution of the poles condensation, it depends critically on the value of parameter γ. If $\gamma \gtrsim 1$ this contribution is very small. For example at $\gamma = 1$, $\xi = \tau = 2.3$, $4\pi^2\gamma\xi/\tau^3 = 7.5 \gg 1$ so that asymptotics (39) is correct. We get $A_1 \sim 0.02$ ($|A_1|^2 \sim 4 \cdot 10^{-4}$) which is smaller than the cut contribution. If γ is small, however, the situation is quite different. For instance, let γ be 0.05. Then $4\pi^2\gamma\xi/\tau^3 = 0.37$ i.e. is smaller than unity. Estimating A_1 by eq. (40) ($m = 1$), we see that $A_1 \sim 60$. The magnitude of cross-section in the peak $\tau = \xi$ five orders larger than the contribution of the cut ~ 100 nb).

It seems to us that two qualitative statements are of special interest. (i) The asymptotics of the cross-section $d\sigma/d\Omega$ ($e^-e^+ \to \mu^-\mu^+$) cannot be smaller than $\sim 1/s$ (we remind that there can exist singularities located to the right of the point $j = 0$ and then the magnitude of the cross-section will be larger). (ii) It is possible (if γ is small) that there exist some rapid variations of the cross section in the region of very small t, say, $s^{-1/2} \gg |t| \gg s^{-1}$ ($\frac{1}{2}\xi \leqslant \tau \leqslant \xi$) whereas the usual range is $(-t) \sim (\ln s)^{-1}$ or $(-t) \sim 1$ (in absence of massless particles).

It has been shown above that for elastic scattering some singularity must exist to the right of the point $j = 0$ but to the left of $j = 1$. In such a situation it seems not unreasonable to suggest the existance of the Pomeranchuk singularity with intercept $\alpha(0) = 1$. One should then expect the constant total cross-section of order of:

$$\sigma_{tot} \sim 4\pi s_0^{-1} \sim G \approx 5 \text{ nb}. \tag{51}$$

We are grateful to A.D. Dolgov, I.T. Dyatlov, L.B. Okun and V.I. Zakharov for numerous useful discussions.

References

[1] G. t'Hooft, Nucl. Phys. B35 (1971) 167;
P.W. Higgs, Phys. Rev. 145 (1966) 1156;
T.W.B. Kibbe, Phys. Rev. 155 (1967) 1554;
S. Weinberg, Phys. Rev. Letters 19 (1967) 1264.

[2] A.D. Dolgov, L.B. Okun and V.I. Zakharov, preprint ITEPH N 877, Moscow (1971).

[3] A.A. Anslem and V.N. Gribov, Phys. Letters 40B (1972) 487.

[4] A.D. Dolgov, V.N. Gribov, L.B. Okun and B.I. Zakharov, in print.

[5] A.D. Dolgov, L.B. Okun and V.I. Zakharov, Phys. Letters 37B (1971) 298.

[6] Ya.I. Azimov, Yad. Fiz. 4 (1966) 663.

[7] Ya.I. Azimov, A.A. Anselm and V.M. Shekhter, JETP (Sov. Phys.) 44 (1963) 361.
[8] V.N. Gribov and I.Ya. Pomeranchuk, JETP (Sov. Phys.) 43 (1962) 308.
[9] V.N. Gribov and I.Ya. Pomeranchuk, JETP (Sov. Phys.) 42 (1962) 1971.
[10] V.N. Gribov and I.Ya. Pomeranchuk, Phys. Rev. Letters 2 (1962) 232.
[11] Ya.I. Azimov, JETP (Sov. Phys.) 43 (1962) 2321.
[12] S. Mandelstam, Nuovo Cimento 30 (1963) 1113, 1127, 1143.
[13] V.N. Gribov, JETP (Sov. Phys.) 53 (1967) 654.
[14] V.N. Gribov, I.Ya. Pomeranchuk and K.A. Ter-Martirosyan, Yad. Fiz. 2 (1965) 361.